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The β-function in duality-covariant non-commutative ϕ^4-theory

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Abstract. We compute the one-loop β-functions describing the renormalisation of the coupling constant λ and the frequency parameter Ω for the real four-dimensional duality-covariant non-commutative ϕ^4-model, which is renormalisable to all orders. The contribution from the one-loop four-point function is reduced by the one-loop wavefunction renormalisation, but the β_λ-function remains non-negative. Both β_λ and β_Ω vanish at the one-loop level for the duality-invariant model characterised by $\Omega = 1$. Moreover, β_Ω also vanishes in the limit $\Omega \to 0$, which defines the standard non-commutative ϕ^4-quantum field theory. Thus, the limit $\Omega \to 0$ exists at least at the one-loop level.

1 Introduction

For many years, the renormalisation of quantum field theories on non-commutative $\mathbb{R}^4$ has been an open problem [1]. Recently, we have proven in [2] that the real duality-covariant ϕ^4-model on non-commutative $\mathbb{R}^4$ is renormalisable to all orders. The duality transformation exchanges positions and momenta [3],

$$\begin{aligned} \hat{\phi}(p) &\leftrightarrow \pi^2 \sqrt{|\det\theta|}\,\phi(x)\,, \\ p_\mu &\leftrightarrow \tilde{x}_\mu := 2\left(\theta^{-1}\right)_{\mu\nu} x^\nu\,, \end{aligned} \tag{1}$$

where $\hat{\phi}(p_a) = \int \mathrm{d}^4x\, \mathrm{e}^{(-1)^a \mathrm{i} p_{a,\mu} x_a^\mu} \phi(x_a)$. The subscript a refers to the cyclic order in the $\star$-product. The duality-covariant non-commutative ϕ^4-action is given by

$$\begin{aligned} &S[\phi;\mu_0,\lambda,\Omega] \\ &:= \int \mathrm{d}^4x \left(\frac{1}{2}\left(\partial_\mu\phi\right)\star\left(\partial^\mu\phi\right) + \frac{\Omega^2}{2}\left(\tilde{x}_\mu\phi\right)\star\left(\tilde{x}^\mu\phi\right) \right. \\ &\qquad \left. + \frac{\mu_0^2}{2}\phi\star\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi \right)(x)\,. \end{aligned} \tag{2}$$

Under the transformation (1) one has

$$S\left[\phi;\mu_0,\lambda,\Omega\right] \mapsto \Omega^2 S\left[\phi;\frac{\mu_0}{\Omega},\frac{\lambda}{\Omega^2},\frac{1}{\Omega}\right]\,. \tag{3}$$

In the special case $\Omega = 1$ the action $S[\phi;\mu_0,\lambda,1]$ is invariant under the duality (1). Moreover, $S[\phi;\mu_0,\lambda,1]$ can be written as a standard matrix model which is closely related to an exactly solvable model [4].

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Knowing that the action (2) gives rise to a renormalisable quantum field theory [2], it is interesting to compute the β_λ- and β_Ω-functions which describe the renormalisation of the coupling constant λ and of the oscillator frequency Ω. Whereas we have proven the renormalisability in the Wilson–Polchinski approach [5, 6] adapted to non-local matrix models [7], we compute the one-loop β_λ- and β_Ω-functions by standard Feynman graph calculations. Of course, these are Feynman graphs parametrised by matrix indices instead of momenta. We rely heavily on the power-counting behaviour proven in [2], which allows us to ignore in the β-functions all non-planar graphs and the detailed index dependence of the planar two- and four-point graphs. Thus, only the lowest-order (discrete) Taylor expansion of the planar two- and four-point graphs can contribute to the β-functions. This means that we cannot refer to the usual symmetry factors of commutative ϕ^4-theory so that we have to carefully recompute the graphs.

We obtain interesting consequences for the limiting cases $\Omega = 1$ and $\Omega = 0$ as discussed in Sect. 5.

2 Definition of the model

The non-commutative $\mathbb{R}^4$ is defined as the algebra $\mathbb{R}^4_\theta$ which as a vector space is given by the space $\mathcal{S}(\mathbb{R}^4)$ of (complex-valued) Schwartz class functions of rapid decay, equipped with the multiplication rule

$$\begin{aligned} (a\star b)(x) &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\int \mathrm{d}^4y\, a\left(x+\frac{1}{2}\theta\cdot k\right) b(x+y)\,\mathrm{e}^{\mathrm{i}k\cdot y}\,, \\ (\theta\cdot k)^\mu &= \theta^{\mu\nu}k_\nu\,, \quad k\cdot y = k_\mu y^\mu\,, \quad \theta^{\mu\nu} = -\theta^{\nu\mu}\,. \end{aligned} \tag{4}$$

We place ourselves into a coordinate system in which the only non-vanishing components $\theta_{\mu\nu}$ are $\theta_{12} = -\theta_{21} =$

$\theta_{34} = -\theta_{42} = \theta$. We use an adapted base,

$$b_{mn}(x) = f_{m^1n^1}\left(x^1, x^2\right) f_{m^2n^2}\left(x^3, x^4\right) , \quad (5)$$
$$m = {}^{m^1}_{m^2} \in \mathbb{N}^2 ,\ n = {}^{n^1}_{n^2} \in \mathbb{N}^2 ,$$

where the base $f_{m^1n^1}(x^1, x^2) \in \mathbb{R}^2_\theta$ is given in [8]. This base satisfies

$$(b_{mn} \star b_{kl})(x) = \delta_{nk} b_{ml}(x) ,$$
$$\int \mathrm{d}^4x\, b_{mn}(x) = 4\pi^2\theta^2\delta_{mn} . \quad (6)$$

According to [2], the duality-covariant ϕ^4-action (2) has an expansion as follows in the matrix base (5):

$$S[\phi; \mu_0, \lambda, \Omega] \quad (7)$$
$$= 4\pi^2\theta^2 \sum_{m,n,k,l\in\mathbb{N}^2} \left(\frac{1}{2} G_{mn;kl}\phi_{mn}\phi_{kl} + \frac{\lambda}{4!}\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\right) ,$$

where $\phi(x) = \sum_{m,n\in\mathbb{N}^2} \phi_{mn} b_{mn}(x)$ and

$$\begin{aligned} G_{mn;kl} = {}& \left(\mu_0^2 + \frac{2}{\theta}\left(1+\Omega^2\right)\left(m^1+n^1+m^2+n^2+2\right)\right) \\ & \times\delta_{n^1k^1}\delta_{m^1l^1}\delta_{n^2k^2}\delta_{m^2l^2} \\ & -\frac{2}{\theta}\left(1-\Omega^2\right)\Big(\Big(\sqrt{(n^1+1)(m^1+1)}\,\delta_{n^1+1,k^1}\delta_{m^1+1,l^1} \\ & \quad +\sqrt{n^1m^1}\,\delta_{n^1-1,k^1}\delta_{m^1-1,l^1}\Big)\delta_{n^2k^2}\delta_{m^2l^2} \\ & +\Big(\sqrt{(n^2+1)(m^2+1)}\,\delta_{n^2+1,k^2}\delta_{m^2+1,l^2} \\ & \quad +\sqrt{n^2m^2}\,\delta_{n^2-1,k^2}\delta_{m^2-1,l^2}\Big)\delta_{n^1k^1}\delta_{m^1l^1}\Big) . \end{aligned} \quad (8)$$

The quantum field theory is defined by the partition function

$$Z[J] \quad (9)$$
$$= \int \left(\prod_{a,b\in\mathbb{N}^2} \mathrm{d}\phi_{ab}\right) \exp\left(-S[\phi] - 4\pi^2\theta^2 \sum_{m,n\in\mathbb{N}^2} \phi_{mn}J_{nm}\right) .$$

For the free theory defined by $\lambda = 0$ in (7), the solution of (9) is given by

$$Z_{\mathrm{free}}[J] \quad (10)$$
$$= Z_{\mathrm{free}}[0] \exp\left(4\pi^2\theta^2 \sum_{m,n,k,l\in\mathbb{N}^2} \frac{1}{2} J_{mn}\Delta_{mn;kl}J_{kl}\right) ,$$

where the propagator Δ is defined as the inverse of the kinetic matrix G:

$$\sum_{k,l\in\mathbb{N}^2} G_{mn;kl}\Delta_{lk;sr} = \sum_{k,l\in\mathbb{N}^2} \Delta_{nm;lk}G_{kl;rs} = \delta_{mr}\delta_{ns} . \quad (11)$$

We have derived the propagator in [2]:

$$\begin{aligned} \Delta_{{}^{m^1}_{m^2}{}^{n^1}_{n^2};{}^{k^1}_{k^2}{}^{l^1}_{l^2}} &= \frac{\theta}{2(1+\Omega)^2}\delta_{m^1+k^1,n^1+l^1}\delta_{m^2+k^2,n^2+l^2} \\ & \times \sum_{v=\frac{|m-l|}{2}}^{\frac{\min(m+l,n+k)}{2}} B\left(1+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\|m\|+\|k\|)-\|v\|, 1+2\|v\|\right) \\ & \times {}_2F_1\left(\begin{matrix} 1+2\|v\| ,\ \frac{\mu_0^2\theta}{8\Omega}-\frac{1}{2}(\|m\|+\|k\|)+\|v\| \\ 2+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(\|m\|+\|k\|)+\|v\| \end{matrix}\middle| \frac{(1-\Omega)^2}{(1+\Omega)^2}\right) \\ & \times \prod_{i=1}^{2} \left(\sqrt{\binom{n^i}{v^i+\frac{n^i-k^i}{2}}\binom{k^i}{v^i+\frac{k^i-n^i}{2}}} \right. \\ & \left. \times \sqrt{\binom{m^i}{v^i+\frac{m^i-l^i}{2}}\binom{l^i}{v^i+\frac{l^i-m^i}{2}}}\left(\frac{1-\Omega}{1+\Omega}\right)^{2v^i}\right) , \end{aligned} \quad (12)$$

where $\sum_{v=a}^{b} := \sum_{v^1=a^1}^{b^1}\sum_{v^2=a^2}^{b^2}$ and $\|a\| := a^1 + a^2$. Here, $B(a, b)$ is the Beta-function and ${}_2F_1\left({a,b \atop c}\middle|z\right)$ the hypergeometric function.

As usual we solve the interacting theory perturbatively:

$$Z[J] = \exp\left(-V\left[\frac{\partial}{\partial J}\right]\right) Z_{\mathrm{free}}[J] , \quad (13)$$
$$V\left[\frac{\partial}{\partial J}\right] := \frac{\lambda}{4!(4\pi^2\theta^2)^3} \sum_{m,n,k,l\in\mathbb{N}^2} \frac{\partial^4}{\partial J_{ml}\,\partial J_{lk}\,\partial J_{kn}\,\partial J_{nm}} .$$

It is convenient to pass to the generating functional of connected Green's functions, $W[J] = \ln Z[J]$:

$$W[J] = W_{\mathrm{free}}[J] \quad (14)$$
$$+ \ln\left(1+\mathrm{e}^{-W_{\mathrm{free}}[J]}\left(\exp\left(-V\left[\frac{\partial}{\partial J}\right]\right)-1\right)\mathrm{e}^{W_{\mathrm{free}}[J]}\right) ,$$

where $W_{\mathrm{free}}[J] := \ln Z_{\mathrm{free}}[J]$. In order to obtain the expansion in λ one has to expand $\ln(1 + x)$ as a power series in x and $\exp(-V)$ as a power series in V. By a Legendre transformation we pass to the generating functional of one-particle irreducible (1PI) Green's functions:

$$\Gamma\left[\phi^{\mathrm{cl}}\right] := 4\pi^2\theta^2 \sum_{m,n\in\mathbb{N}^2} \phi^{\mathrm{cl}}_{mn}J_{nm} - W[J] , \quad (15)$$

where J has to be replaced by the inverse solution of

$$\phi^{\mathrm{cl}}_{mn} := \frac{1}{4\pi^2\theta^2}\frac{\partial W[J]}{\partial J_{nm}} . \quad (16)$$

3 Renormalisation group equation

The computation of the expansion coefficients

$$\Gamma_{m_1n_1;\dots;m_Nn_N} := \frac{1}{N!}\frac{\partial^N\Gamma\left[\phi^{\mathrm{cl}}\right]}{\partial\phi^{\mathrm{cl}}_{m_1n_1}\dots\partial\phi^{\mathrm{cl}}_{m_Nn_N}} \quad (17)$$

of the effective action involves possibly divergent sums over undetermined loop indices. Therefore, we have to introduce a cut-off $\mathcal{N}$ for all loop indices. According to [2], the expansion coefficients (17) can be decomposed into a relevant/marginal and an irrelevant piece. As a result of the renormalisation proof, the relevant/marginal parts have – after a rescaling of the field amplitude – the same form as the initial action (2), (7) and (8), now parametrised by the "physical" mass, coupling constant and oscillator frequency:

$$\Gamma_{\mathrm{rel/marg}}\left[\mathcal{Z}\phi^{\mathrm{cl}}\right] = S\left[\phi^{\mathrm{cl}};\mu_{\mathrm{phys}},\lambda_{\mathrm{phys}},\Omega_{\mathrm{phys}}\right]\,. \tag{18}$$

In the renormalisation process, the physical quantities μ^2_{phys}, λ_{phys} and Ω_{phys} are kept constant with respect to the cut-off $\mathcal{N}$. This is achieved by starting from a carefully adjusted initial action $S\left[\mathcal{Z}[\mathcal{N}]\phi,\mu_0[\mathcal{N}],\lambda[\mathcal{N}],\Omega[\mathcal{N}]\right]$, which gives rise to the bare effective action $\Gamma[\phi^{\mathrm{cl}};\mu_0[\mathcal{N}],\lambda[\mathcal{N}],\Omega[\mathcal{N}],\mathcal{N}]$. Expressing the bare parameters μ_0,λ,Ω as a function of the physical quantities and the cut-off, the expansion coefficients of the renormalised effective action

$$\begin{aligned}&\Gamma^R\left[\phi^{\mathrm{cl}};\mu_{\mathrm{phys}},\lambda_{\mathrm{phys}},\Omega_{\mathrm{phys}}\right] \qquad (19)\\ &:= \Gamma\left[\mathcal{Z}[\mathcal{N}]\phi^{\mathrm{cl}},\mu_0[\mathcal{N}],\lambda[\mathcal{N}],\Omega[\mathcal{N}],\mathcal{N}\right]\Big|_{\substack{\mu_{\mathrm{phys}}=\mathrm{const}\\ \lambda_{\mathrm{phys}}=\mathrm{const}\\ \Omega_{\mathrm{phys}}=\mathrm{const}}}\end{aligned}$$

are finite and convergent in the limit $\mathcal{N}\to\infty$. In other words,

$$\begin{aligned}&\lim_{\mathcal{N}\to\infty}\mathcal{N}\frac{\mathrm{d}}{\mathrm{d}\mathcal{N}}\left(\mathcal{Z}^N[\mathcal{N}]\right. \qquad (20)\\ &\left.\times\,\Gamma_{m_1n_1;\dots;m_Nn_N}[\mu_0[\mathcal{N}],\lambda[\mathcal{N}],\Omega[\mathcal{N}],\mathcal{N}]\right)=0\,.\end{aligned}$$

This implies the renormalisation group equation

$$\begin{aligned}&\lim_{\mathcal{N}\to\infty}\left(\mathcal{N}\frac{\partial}{\partial\mathcal{N}}+N\gamma+\mu_0^2\beta_{\mu_0}\frac{\partial}{\partial\mu_0^2}+\beta_\lambda\frac{\partial}{\partial\lambda}+\beta_\Omega\frac{\partial}{\partial\Omega}\right)\\ &\times\,\Gamma_{m_1n_1;\dots;m_Nn_N}[\mu_0,\lambda,\Omega,\mathcal{N}]=0\,, \qquad (21)\end{aligned}$$

where

$$\beta_{\mu_0}=\frac{1}{\mu_0^2}\mathcal{N}\frac{\partial}{\partial\mathcal{N}}\left(\mu_0^2[\mu_{\mathrm{phys}},\lambda_{\mathrm{phys}},\Omega_{\mathrm{phys}},\mathcal{N}]\right)\,, \tag{22}$$

$$\beta_\lambda=\mathcal{N}\frac{\partial}{\partial\mathcal{N}}\left(\lambda[\mu_{\mathrm{phys}},\lambda_{\mathrm{phys}},\Omega_{\mathrm{phys}},\mathcal{N}]\right)\,, \tag{23}$$

$$\beta_\Omega=\mathcal{N}\frac{\partial}{\partial\mathcal{N}}\left(\Omega[\mu_{\mathrm{phys}},\lambda_{\mathrm{phys}},\Omega_{\mathrm{phys}},\mathcal{N}]\right)\,, \tag{24}$$

$$\gamma=\mathcal{N}\frac{\partial}{\partial\mathcal{N}}\left(\ln\mathcal{Z}[\mu_{\mathrm{phys}},\lambda_{\mathrm{phys}},\Omega_{\mathrm{phys}},\mathcal{N}]\right)\,. \tag{25}$$

4 One-loop computations

Defining $(\Delta J)_{mn}:=\sum_{p,q\in\mathbb{N}^2}\Delta_{mn;pq}J_{pq}$ we write (parts of) the generating functional of connected Green's functions up to second order in λ:

$$\begin{aligned}
W[J]&=\ln Z[0]+4\pi^2\theta^2\sum_{m,n,k,l\in\mathbb{N}^2}\frac{1}{2}J_{mn}\Delta_{mn;kl}J_{kl}\\
&-\left(4\pi^2\theta^2\right)\frac{\lambda}{4!}\sum_{m,n,k,l\in\mathbb{N}^2}\Big\{(\Delta J)_{ml}(\Delta J)_{lk}(\Delta J)_{kn}(\Delta J)_{nm}\\
&\quad+\frac{1}{4\pi^2\theta^2}\big(\Delta_{nm;kn}(\Delta J)_{ml}(\Delta J)_{lk}\\
&\qquad+\Delta_{kn;lk}(\Delta J)_{nm}(\Delta J)_{ml}+\Delta_{nm;ml}(\Delta J)_{lk}(\Delta J)_{kn}\\
&\qquad+\Delta_{lk;ml}(\Delta J)_{kn}(\Delta J)_{nm}\big)\\
&\quad+\frac{1}{4\pi^2\theta^2}\big(\Delta_{nm;lk}(\Delta J)_{kn}(\Delta J)_{ml}\\
&\qquad+\Delta_{kn;ml}(\Delta J)_{nm}(\Delta J)_{lk}\big)\\
&+\frac{1}{(4\pi^2\theta^2)^2}\big((\Delta_{nm;kn}\Delta_{lk;ml}+\Delta_{kn;lk}\Delta_{nm;ml})\\
&+\Delta_{nm;lk}\Delta_{kn;ml}\big)\Big\}\\
&+\frac{\lambda^2}{2(4!)^2}\sum_{m,n,k,l,r,s,t,u\in\mathbb{N}^2}\Big\{\Big[\big(\Delta_{ml;sr}\Delta_{lk;ts}(\Delta J)_{kn}(\Delta J)_{nm}\\
&\quad+\Delta_{ml;sr}\Delta_{kn;ts}(\Delta J)_{lk}(\Delta J)_{nm}\\
&\quad+\Delta_{ml;sr}\Delta_{nm;ts}(\Delta J)_{lk}(\Delta J)_{kn}\\
&\quad+\Delta_{lk;sr}\Delta_{ml;ts}(\Delta J)_{kn}(\Delta J)_{nm}\\
&\quad+\Delta_{lk;sr}\Delta_{kn;ts}(\Delta J)_{ml}(\Delta J)_{nm}\\
&\quad+\Delta_{lk;sr}\Delta_{nm;ts}(\Delta J)_{ml}(\Delta J)_{kn}\\
&\quad+\Delta_{kn;sr}\Delta_{ml;ts}(\Delta J)_{lk}(\Delta J)_{nm}\\
&\quad+\Delta_{kn;sr}\Delta_{lk;ts}(\Delta J)_{ml}(\Delta J)_{nm}\\
&\quad+\Delta_{kn;sr}\Delta_{nm;ts}(\Delta J)_{ml}(\Delta J)_{lk}\\
&\quad+\Delta_{nm;sr}\Delta_{ml;ts}(\Delta J)_{lk}(\Delta J)_{kn}\\
&\quad+\Delta_{nm;sr}\Delta_{lk;ts}(\Delta J)_{ml}(\Delta J)_{kn}\\
&\quad+\Delta_{nm;sr}\Delta_{kn;ts}(\Delta J)_{ml}(\Delta J)_{lk}\big)(\Delta J)_{ru}(\Delta J)_{ut}\\
&\quad+5\text{ permutations of }{}_{ts,\ sr,\,ru,\,ut}\Big]\\
&\quad+\text{1PI-contributions with }\leq 2\ J\text{'s}\\
&\quad+\text{1PR-contributions}\Big\}+\mathcal{O}\left(\lambda^3\right)\,. \qquad (26)
\end{aligned}$$

In second order in λ we get a huge number of terms so that we display only the 1PI-contribution with four J's.

For the classical field (16) we get

$$\phi^{\mathrm{cl}}_{mn}=\sum_{p,q\in\mathbb{N}^2}\Delta_{nm;pq}J_{pq}+\mathcal{O}(\lambda)$$

so that

$$J_{pq}=\sum_{r,s\in\mathbb{N}^2}G_{qp;rs}\phi^{\mathrm{cl}}_{rs}+\mathcal{O}(\lambda)\,. \tag{27}$$

The remaining part not displayed in (27) removes the 1PR-contributions when passing to $\Gamma\left[\phi^{\mathrm{cl}}\right]$. We thus obtain

$$
\begin{aligned}
\Gamma\left[\phi^{\mathrm{cl}}\right] = \Gamma[0] + 4\pi^2\theta^2 \sum_{m,n,k,l\in\mathbb{N}^2} \frac{1}{2} \Bigg\{ \Bigg(& G_{mn;kl} \\
& + \frac{\lambda}{6\left(4\pi^2\theta^2\right)} \sum_{p\in\mathbb{N}^2} \left(\delta_{ml}\Delta_{pn;kp} + \delta_{kn}\Delta_{mp;pl}\right)\Bigg) && \text{(28a)} \\
& + \frac{\lambda}{6\left(4\pi^2\theta^2\right)} \Delta_{ml;kn} && \text{(28b)} \\
& + \mathcal{O}\left(\lambda^2\right) \Bigg\} \phi^{\mathrm{cl}}_{mn}\phi^{\mathrm{cl}}_{kl} \\
+ 4\pi^2\theta^2 \sum_{m,n,k,l,r,s,t,u\in\mathbb{N}^2} \frac{\lambda}{4!} \Bigg\{ & \delta_{nk}\delta_{lr}\delta_{st}\delta_{um} && \text{(28c)} \\
& - \frac{\lambda}{2(4!)\left(4\pi^2\theta^2\right)} \Bigg(\sum_{p,q\in\mathbb{N}^2} \big(4\Delta_{mp;qs}\Delta_{pl;tq}\delta_{kn}\delta_{ur} \\
& \quad + 4\Delta_{kp;qs}\Delta_{pn;tq}\delta_{ml}\delta_{ur} + 4\Delta_{pl;rq}\Delta_{mp;qu}\delta_{nk}\delta_{st} \\
& \quad + 4\Delta_{pn;rq}\Delta_{kp;qu}\delta_{ml}\delta_{st}\big) && \text{(28d)} \\
& + \sum_{p\in\mathbb{N}^2} \big(4\Delta_{ml;ps}\Delta_{kn;tp}\delta_{ur} + 4\Delta_{kn;ps}\Delta_{ml;tp}\delta_{ur} \\
& \quad + 4\Delta_{mp;ts}\Delta_{pl;ru}\delta_{nk} + 4\Delta_{pl;ts}\Delta_{mp;ru}\delta_{nk} \\
& \quad + 4\Delta_{kp;ts}\Delta_{pn;ru}\delta_{ml} + 4\Delta_{pn;ts}\Delta_{kp;ru}\delta_{ml} \\
& \quad + 4\Delta_{ml;rp}\Delta_{kn;pu}\delta_{st} \\
& \quad + 4\Delta_{kn;rp}\Delta_{ml;pu}\delta_{st}\big) && \text{(28e)} \\
& + \sum_{p,q\in\mathbb{N}^2} \big(4\Delta_{pl;qs}\Delta_{mp;tq}\delta_{nk}\delta_{ur} \\
& \quad + 4\Delta_{pn;qs}\Delta_{kp;tq}\delta_{ml}\delta_{ur} + 4\Delta_{kp;rq}\Delta_{pn;qu}\delta_{ml}\delta_{st} \\
& \quad + 4\Delta_{mp;rq}\Delta_{pl;qu}\delta_{nk}\delta_{st}\big) && \text{(28f)} \\
& + 4\Delta_{ml;ts}\Delta_{kn;ru} + 4\Delta_{kn;ts}\Delta_{ml;ru}\Bigg) && \text{(28g)} \\
& + \mathcal{O}\left(\lambda^2\right) \Bigg\} \phi^{\mathrm{cl}}_{mn}\phi^{\mathrm{cl}}_{kl}\phi^{\mathrm{cl}}_{rs}\phi^{\mathrm{cl}}_{tu} + \mathcal{O}\left(\left(\phi^{\mathrm{cl}}\right)^6\right) .
\end{aligned}
$$

Here, (28a) contains the contribution to the planar two-point function and (28b) the contribution to the non-planar two-point function. Next, (28c) and (28d) contribute to the planar four-point function, whereas (28e), (28f) and (28g) constitute three different types of non-planar four-point functions.

Introducing the cut-off $p^i, q^i \leq \mathcal{N}$ in the internal sums over $p, q \in \mathbb{N}^2$, we split the effective action according to [2] as follows into a relevant/marginal and an irrelevant piece ($\Gamma[0]$ can be ignored):

$$
\Gamma\left[\phi^{\mathrm{cl}}\right] \equiv \Gamma_{\mathrm{rel/marg}}\left[\phi^{\mathrm{cl}}\right] + \Gamma_{\mathrm{irrel}}\left[\phi^{\mathrm{cl}}\right] , \tag{29}
$$

with

$$
\begin{aligned}
\Gamma_{\mathrm{rel/marg}}\left[\phi^{\mathrm{cl}}\right] = 4\pi^2\theta^2 \sum_{m,n,k,l\in\mathbb{N}^2} \frac{1}{2} \Bigg\{ & G_{mn;kl} \\
& + \frac{\lambda}{6\left(4\pi^2\theta^2\right)} \delta_{ml}\delta_{kn} \Bigg(2 \sum_{p^1,p^2=0}^{N} \Delta_{\substack{0p^1\\0p^2};\substack{p^10\\p^20}} \\
& \quad + (m^1+n^1+m^2+n^2) \\
& \quad \times \sum_{p^1,p^2=0}^{N} \Bigg(\Delta_{\substack{1p^1\\0p^2};\substack{p^11\\p^20}} - \Delta_{\substack{0p^1\\0p^2};\substack{p^10\\p^20}} \Bigg)\Bigg) \\
& + \mathcal{O}\left(\lambda^2\right) \Bigg\} \phi^{\mathrm{cl}}_{mn}\phi^{\mathrm{cl}}_{kl} \\
+ 4\pi^2\theta^2 & \\
\times \sum_{m,n,k,l\in\mathbb{N}^2} \frac{\lambda}{4!} \Bigg\{ & 1 - \frac{\lambda}{3\left(4\pi^2\theta^2\right)} \sum_{p^1,p^2=0}^{N} \Bigg(\Delta_{\substack{0p^1\\0p^2};\substack{p^10\\p^20}} \Bigg)^2 \\
& + \mathcal{O}\left(\lambda^2\right) \Bigg\} \phi^{\mathrm{cl}}_{mn}\phi^{\mathrm{cl}}_{nk}\phi^{\mathrm{cl}}_{kl}\phi^{\mathrm{cl}}_{lm} .
\end{aligned} \tag{30}
$$

To the marginal four-point function and the relevant two-point function only the projections to planar graphs with vanishing external indices contribute. The marginal two-point function is given by the next-to-leading term in the discrete Taylor expansion around vanishing external indices.

In a regime where $\lambda[\mathcal{N}]$ is so small that the perturbative expansion is valid in (30), the irrelevant part Γ_{irrel} can be completely ignored. Comparing (30) with the initial action according to (2), (7) and (8), we have $\Gamma_{\mathrm{rel/marg}}[\mathcal{Z}\phi^{\mathrm{cl}}] = S\left[\phi^{\mathrm{cl}}; \mu_{\mathrm{phys}}, \lambda_{\mathrm{phys}}, \Omega_{\mathrm{phys}}\right]$ with

$$
\begin{aligned}
\mathcal{Z} = 1 - \frac{\lambda}{192\pi^2\theta} \sum_{p^1,p^2=0}^{\mathcal{N}} & \Bigg(\Delta_{\substack{1p^1\\0p^2};\substack{p^11\\p^20}} - \Delta_{\substack{0p^1\\0p^2};\substack{p^10\\p^20}} \Bigg) \\
& + \mathcal{O}\left(\lambda^2\right) ,
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
\mu^2_{\mathrm{phys}} = \mu_0^2 \Bigg(1 & \\
& + \frac{\lambda}{12\pi^2\theta^2\mu_0^2} \sum_{p^1,p^2=0}^{\mathcal{N}} \Bigg(2\Delta_{\substack{0p^1\\0p^2};\substack{p^10\\p^20}} - \Delta_{\substack{1p^1\\0p^2};\substack{p^11\\p^20}} \Bigg) \\
& - \frac{\lambda}{96\pi^2\theta} \sum_{p^1,p^2=0}^{\mathcal{N}} \Bigg(\Delta_{\substack{1p^1\\0p^2};\substack{p^11\\p^20}} - \Delta_{\substack{0p^1\\0p^2};\substack{p^10\\p^20}} \Bigg) \\
& + \mathcal{O}\left(\lambda^2\right) \Bigg) ,
\end{aligned} \tag{32}
$$

$$\lambda_{\rm phys} = \lambda\Bigg(1 - \frac{\lambda}{12\pi^2\theta^2}\sum_{p^1,p^2=0}^{\mathcal{N}}\Big(\Delta_{\substack{0p^1\\0p^2};\substack{p^10\\p^20}}\Big)^2 - \frac{\lambda}{48\pi^2\theta}\sum_{p^1,p^2=0}^{\mathcal{N}}\Big(\Delta_{\substack{1p^1\\0p^2};\substack{p^11\\p^20}} - \Delta_{\substack{0p^1\\0p^2};\substack{p^10\\p^20}}\Big) + \mathcal{O}\left(\lambda^2\right)\Bigg)\,, \tag{33}$$

$$\Omega_{\rm phys} = \Omega\Bigg(1 + \frac{\lambda\left(1-\Omega^2\right)}{192\pi^2\theta\Omega^2}\sum_{p^1,p^2=0}^{\mathcal{N}}\Big(\Delta_{\substack{1p^1\\0p^2};\substack{p^11\\p^20}} - \Delta_{\substack{0p^1\\0p^2};\substack{p^10\\p^20}}\Big) + \mathcal{O}\left(\lambda^2\right)\Bigg)\,. \tag{34}$$

Solving (32), (33) and (34) for the bare quantities, we obtain to one-loop order

$$\begin{aligned}
&\mu_0^2\left[\mu_{\rm phys},\lambda_{\rm phys},\Omega_{\rm phys},\mathcal{N}\right]\\
&= \mu^2_{\rm phys}\Bigg(1 - \frac{\lambda_{\rm phys}}{12\pi^2\theta^2\mu^2_{\rm phys}}\sum_{p^1,p^2=0}^{\mathcal{N}}\Delta_{\substack{0p^1\\0p^2};\substack{p^10\\p^20}} + \frac{\lambda_{\rm phys}}{96\pi^2\theta}\left(1+\frac{8}{\theta\mu^2_{\rm phys}}\right)\\
&\quad\times\sum_{p^1,p^2=0}^{\mathcal{N}}\Big(\Delta_{\substack{1p^1\\0p^2};\substack{p^11\\p^20}} - \Delta_{\substack{0p^1\\0p^2};\substack{p^10\\p^20}}\Big) + \mathcal{O}\left(\lambda^2_{\rm phys}\right)\Bigg)\,,
\end{aligned} \tag{35}$$

$$\begin{aligned}
&\lambda\left[\mu_{\rm phys},\lambda_{\rm phys},\Omega_{\rm phys},\mathcal{N}\right]\\
&= \lambda_{\rm phys}\Bigg(1 + \frac{\lambda_{\rm phys}}{12\pi^2\theta^2}\sum_{p^1,p^2=0}^{\mathcal{N}}\Big(\Delta_{\substack{0p^1\\0p^2};\substack{p^10\\p^20}}\Big)^2 + \frac{\lambda_{\rm phys}}{48\pi^2\theta}\sum_{p^1,p^2=0}^{\mathcal{N}}\Big(\Delta_{\substack{1p^1\\0p^2};\substack{p^11\\p^20}} - \Delta_{\substack{0p^1\\0p^2};\substack{p^10\\p^20}}\Big) + \mathcal{O}\left(\lambda^2_{\rm phys}\right)\Bigg)\,,
\end{aligned} \tag{36}$$

$$\begin{aligned}
&\Omega\left[\mu_{\rm phys},\lambda_{\rm phys},\Omega_{\rm phys},\mathcal{N}\right]\\
&= \Omega_{\rm phys}\Bigg(1 - \frac{\lambda_{\rm phys}\left(1-\Omega^2_{\rm phys}\right)}{192\pi^2\theta\Omega^2_{\rm phys}}\times\sum_{p^1,p^2=0}^{\mathcal{N}}\Big(\Delta_{\substack{1p^1\\0p^2};\substack{p^11\\p^20}} - \Delta_{\substack{0p^1\\0p^2};\substack{p^10\\p^20}}\Big) + \mathcal{O}\left(\lambda^2_{\rm phys}\right)\Bigg)\,.
\end{aligned} \tag{37}$$

Inserting (12) into (36) we can now compute the β_λ-function (23) up to one-loop order, omitting the index $_{\rm phys}$ on μ^2 and Ω for simplicity:

$$\begin{aligned}
\beta_\lambda = \frac{\lambda^2_{\rm phys}}{48\pi^2}\mathcal{N}\frac{\partial}{\partial\mathcal{N}}\sum_{p^1,p^2=0}^{\mathcal{N}}\Bigg\{
&\left(\frac{{}_2F_1\left(\begin{matrix}1\,,\ \frac{\mu_0^2\theta}{8\Omega}-\frac{1}{2}(p^1+p^2)\\ 2+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2)\end{matrix}\Big|\frac{(1-\Omega)^2}{(1+\Omega)^2}\right)}{(1+\Omega)^2\left(1+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2)\right)}\right)^2\\
&+\frac{p^1(1-\Omega)^2\,{}_2F_1\left(\begin{matrix}3\,,\ \frac{1+\mu_0^2\theta}{8\Omega}-\frac{1}{2}(p^1+p^2+1)\\ 3+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2+1)\end{matrix}\Big|\frac{(1-\Omega)^2}{(1+\Omega)^2}\right)}{(1+\Omega)^4\prod_{s=0}^2\left(\frac{1+2s}{2}+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2)\right)}\\
&+\left(\frac{{}_2F_1\left(\begin{matrix}1\,,\ \frac{\mu_0^2\theta}{8\Omega}-\frac{1}{2}(p^1+p^2+1)\\ 2+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2+1)\end{matrix}\Big|\frac{(1-\Omega)^2}{(1+\Omega)^2}\right)}{2(1+\Omega)^2\left(\frac{3}{2}+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2)\right)}\right.\\
&\left.-\frac{{}_2F_1\left(\begin{matrix}1\,,\ \frac{\mu_0^2\theta}{8\Omega}-\frac{1}{2}(p^1+p^2)\\ 2+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2)\end{matrix}\Big|\frac{(1-\Omega)^2}{(1+\Omega)^2}\right)}{2(1+\Omega)^2\left(1+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(p^1+p^2)\right)}\right)\\
&+\mathcal{O}(\lambda_{\rm phys})\Bigg\}\,.
\end{aligned} \tag{38}$$

Symmetrising the numerator in the third line $p^1 \mapsto \frac{1}{2}(p^1+p^2)$ and using the expansions

$$\begin{aligned}
&{}_2F_1\left(\begin{matrix}1\,,\ a-p\\ b+p\end{matrix}\Big|z\right)\\
&= \frac{1}{1+z} + \frac{z(a+b)+z^2(a+b-2)}{p(1+z)^3} + \mathcal{O}(p^{-2})\,,\\
&{}_2F_1\left(\begin{matrix}3\,,\ a-p\\ b+p\end{matrix}\Big|z\right) = \frac{1}{(1+z)^3} + \mathcal{O}(p^{-1})\,,
\end{aligned} \tag{39}$$

which are valid for large p, we obtain up to irrelevant contributions vanishing in the limit $\mathcal{N}\to\infty$

$$\beta_\lambda = \frac{\lambda^2_{\text{phys}}}{48\pi^2}\mathcal{N}\frac{\partial}{\partial\mathcal{N}}\sum_{p^1,p^2=0}^{\mathcal{N}}\frac{1}{\left(1+\Omega^2_{\text{phys}}\right)^2}\frac{1}{(1+p^1+p^2)^2}$$
$$\times\left\{1+\frac{\left(1-\Omega^2_{\text{phys}}\right)^2}{2\left(1+\Omega^2_{\text{phys}}\right)}-\frac{\left(1+\Omega^2_{\text{phys}}\right)}{2}\right\}$$
$$+\mathcal{O}\left(\lambda^3_{\text{phys}}\right)+\mathcal{O}\left(\mathcal{N}^{-1}\right)$$
$$= \frac{\lambda^2_{\text{phys}}}{48\pi^2}\frac{\left(1-\Omega^2_{\text{phys}}\right)}{\left(1+\Omega^2_{\text{phys}}\right)^3}+\mathcal{O}\left(\lambda^3_{\text{phys}}\right)+\mathcal{O}\left(\mathcal{N}^{-1}\right) . \quad (40)$$

Similarly, one obtains

$$\beta_\Omega = \frac{\lambda_{\text{phys}}\Omega_{\text{phys}}}{96\pi^2}\frac{\left(1-\Omega^2_{\text{phys}}\right)}{\left(1+\Omega^2_{\text{phys}}\right)^3}+\mathcal{O}\left(\lambda^2_{\text{phys}}\right)+\mathcal{O}\left(\mathcal{N}^{-1}\right) , \quad (41)$$

$$\beta_{\mu_0} = -\frac{\lambda_{\text{phys}}}{48\pi^2\theta\mu^2_{\text{phys}}\left(1+\Omega^2_{\text{phys}}\right)}$$
$$\times\left(4\mathcal{N}\ln(2)+\frac{\left(8+\theta\mu^2_{\text{phys}}\right)\Omega^2_{\text{phys}}}{\left(1+\Omega^2_{\text{phys}}\right)^2}\right)$$
$$+\mathcal{O}\left(\lambda^2_{\text{phys}}\right)+\mathcal{O}\left(\mathcal{N}^{-1}\right) , \quad (42)$$

$$\gamma = \frac{\lambda_{\text{phys}}}{96\pi^2}\frac{\Omega^2_{\text{phys}}}{\left(1+\Omega^2_{\text{phys}}\right)^3}+\mathcal{O}\left(\lambda^2_{\text{phys}}\right)+\mathcal{O}\left(\mathcal{N}^{-1}\right) . \quad (43)$$

5 Discussion

We have computed the one-loop β- and γ-functions in real four-dimensional duality-covariant non-commutative ϕ^4-theory. Remarkably, this model has a one-loop contribution to the wavefunction renormalisation which partly compensates the contribution from the planar one-loop four-point function to the β_λ-function. The one-loop β_λ-function is non-negative and vanishes in the distinguished case $\Omega = 1$ of the duality-invariant model; see (3). At $\Omega = 1$ also the β_Ω-function vanishes. This is of course expected (to all orders), because for $\Omega = 1$ the propagator (12) is diagonal,

$$\Delta_{\substack{m^1n^1;k^1l^1\\m^2n^2;k^2l^2}}\big|_{\Omega=1} = \frac{\delta_{m^1l^1}\delta_{k^1n^1}\delta_{m^2l^2}\delta_{k^2n^2}}{\mu_0^2+(4/\theta)(m^1+m^2+n^1+n^2+2)},$$

so that the Feynman graphs never generate terms with $|m^i-l^i| = |n^i-k^i| = 1$ in (8).

The similarity of the duality-invariant theory with the exactly solvable models discussed in [4] suggests that also the β_λ-function vanishes to all orders for $\Omega = 1$. The crucial differences between our model with $\Omega = 1$ and [4] is that we are using *real* fields, for which it is not so clear that the construction of [4] can be applied. But the planar graphs of a real and a complex ϕ^4-model are very similar, so that we expect identical β_λ-functions (possibly up to a global factor) for the complex and the real model. Since a main feature of [4] was the independence on the dimension of the space, the model with $\Omega = 1$ and matrix cut-off $\mathcal{N}$ should be (more or less) equivalent to a two-dimensional model, which has a mass renormalisation only [8]. Therefore, we conjecture a vanishing β_λ-function in four-dimensional duality-invariant non-commutative ϕ^4-theory to all orders.

The most surprising result is that the one-loop β_Ω-function also vanishes for $\Omega \to 0$. We cannot directly set $\Omega = 0$, because the hypergeometric functions in (38) become singular and the expansions (39) are not valid. Moreover, the power-counting theorems of [2], which we used to project to the relevant/marginal part of the effective action (30), also require $\Omega > 0$. However, in the same way as in the renormalisation of two-dimensional non-commutative ϕ^4-theory [8], it is possible to switch off Ω very weakly with the cut-off $\mathcal{N}$, e.g. with

$$\Omega = \mathrm{e}^{-(\ln(1+\ln(1+\mathcal{N})))^2} . \quad (44)$$

The decay (44) for large $\mathcal{N}$ over-compensates the growth of any polynomial in $\ln\mathcal{N}$, which according to [2] is the bound for the graphs contributing to a renormalisation of Ω. On the other hand, (44) does not modify the expansions (39). Thus, in the limit $\mathcal{N} \to \infty$, we have constructed the usual non-commutative ϕ^4-theory given by $\Omega = 0$ in (2) at the one-loop level. It would be very interesting to know whether this construction of the non-commutative ϕ^4-theory as the limit of a sequence (44) of duality-covariant ϕ^4-models can be extended to higher loop order.

We also notice that the one-loop β_λ- and β_Ω-functions are independent of the non-commutativity scale θ. There is, however, a contribution to the one-loop mass renormalisation via the dimensionless quantity $\mu^2_{\text{phys}}\theta$; see (42).

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